On the Banach-Stone Theorem and the Manifold Topological Classification

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Abstract:

We present a simple set-theoretic proof of the Banach-Stone Theorem. We thus apply this Topological Classification theorem to the still-unsolved problem of topological classification of Euclidean Manifolds through two conjectures and additionaly we give a straightforward proof of the famous Brower theorem for manifolds topologically classified by their Euclidean dimensions.

We start our comment by announcing the:

Banach-Stone Theorem ([1]). Let X and Y be compact Hausdorff spaces, such that the associated function algebras of continuous functions $C(X,R)$ and $C(Y,R)$ separate points in X and Y respectively. We have thus

a) X and Y are homeomorphic $\Leftrightarrow$

b) $C(X,R)$ and $C(Y,R)$ are isomorphic.

Proof (Elementary).

Let us first prove that b) $\Rightarrow$ a). We thus consider the explicitly given isomorphism:

$$\begin{aligned} I\colon C(X,R) &\to C(Y,R) \\ f &\to I(f) \end{aligned} \tag{1}$$

Associate to it, we consider the multiplicative linear continuous functional on $C(Y,R)$ defined below for each point $x \in X$ (fixed)

$$\mathcal{L}(g) \equiv I^{-1}(g)(x). \tag{2}$$

Since $\mathcal{L}(g)$ is multiplicative linear function. We have by a simple application of the Riesz-Markov theorem, that there is a Dirac ($\overline{\mu}$-point supported) measure on Y, such that ([2]) for any continuous function on Y

$$\mathcal{L}(g) = \int_Y g(y) d\,\overline{\mu}(y - \overline{y}) = g(\overline{y}). \tag{3}$$

We have thus constructed our obvious candidate for our homeomorphism between X and Y, namely:

$$\begin{aligned} i\colon X &\to Y \\ x &\to \overline{y}. \end{aligned} \tag{4}$$

Now we can see that i is a function with domain X and range Y, since in the case of existence of two distincts points $\overline{y}_1$ and $\overline{y}_2$ supposedly image of an unique point $x \in X$, certainly for all function $g \in C(Y, R)$, we would have the result

$$I^{-1}(g)(x) = g(\overline{y}_1) = g(\overline{y}_2) \tag{5}$$

which is in clear contradiction with the hypothesis that the function algebras $C(X, R)$ and $C(Y, R)$ separate points. The function i defined by eq.(4) is thus clearly injective since if there is points x_1 and x_2 , with $x_1 \neq x_2$ such that $i(x_1) = i(x_2) = \overline{y}$, naturally this leads again to the contradictory result on point separation of the algebra $C(X, R)$. Namely

$$I^{-1}(g)(x_1) = I^{-1}(g)(x_2) = \overline{y} \tag{6}$$

valid for all functions $I^{-1}(g)$ in $C(X, R)$ since J is supposed to be a isomorphism between the functions algebras $C(X, R)$ and $C(Y, R)$ and both are supposed to separate points (as in the case of Regular Topological spaces for instance!).

By just considering the inverse isomorphism

$$I^{-1}\colon C(Y, R) \to C(X, R) \tag{7}$$

we obtains that i is an onto application between X and Y.

To check the continuity of the application i one can easily apply ultra-filters arguments (in the case of X and Y possessing on enumerable basis it is straightforward to apply the sequential criterium for continuity!).

As a result $i\colon X \to Y$ as defined by eq.(4) defines a homeomorphism between X and Y, that a) $\Rightarrow$ b) is trivial. ■

Let us use the above Topological Classification Theorem to show that if two continuous manifolds M and N with dimensions m and n respectively are homeomorphic, then their dimensions are equal $m = n$. By using charts (simplexs) one reduces the above claimed result to show that if $C([0,1]^n, R)$ is isomorphic to $C([0,1]^m, R)$. $\big[$ Here $[0,1]^n$ and $[0,1]^m$ are n and m dimensional cubes (simplexs)$\big]$, then $n = m$ (the Brower dimension theorem), since the set of coordinate projections make closed sub-algebras on the above considered continuous functions $C([0,1]^n, R)$ and $C([0,1]^m, R)$, by a straightforward application of Weierstrass-Stone theorem ([3]), we have the famous Brower Topological Dimension Theorem as a scolium of the combined Stone-Banach theorem.

Let us conjecture the following generalization of the Banach-Stone of ours to the case of manifolds possessing a C^k-differentiable structure.

Conjecture 1. *M and N are C^k-diffeomorphics if and only if $C^k(M)$ and $C^k(N)$ are isomorphics as functions algebras.*

Another dimensional reduction procedure to classify topologically-differentiable C^k-manifolds is the following (for $k \geq 1$).

Firstly we introduce some "surgery-tomographic operations" on the given manifold. Let $T_x(M)$ be the tangent space of M at the point $x \in M$. We consider thus the Whitney-Nash manifold "ambient" immersion euclidean space R^{2m+1} ([3]), namely $M \hookrightarrow R^{2m+1}$. We thus "rotate" the tangent plane $T_x(M)$ in all directions "$\mathcal{O}$" of the ambient immersions euclidean space R^{2m+1}: $R_{\mathcal{O}}\, T_x(M)$. These are obviously m-dimensional affim euclidean spaces. We further suppose that $\{R_{\mathcal{O}}\, T_x(M) \cap M\} \equiv M_{\mathcal{O}}^{(m-1)}(x)$ are all C^k-manifolds of dimension less than m (for definiteness $m-1$). We make the additional hypothesis that there is a $1-1$, onto and continuous function of M into N.

Conjecture 2. *Then if M is homeomorphic topologically to N. $M_{\mathcal{O}}^{(m-1)}(x)$ is homeomorphic topologically to $N_{\mathcal{O}}^{(m-1)}(f(x))$ and the converse is true for all "directions" θ and all $x \in M$.*

Disclaimer: This author Luiz C.L. Botelho althought Full Professor at the Mathematical Institute of Fluminense Federal University, DOES NOT PARTICIPATE OR BELONGS TO ITS POST-GRADUATE RESEARCH GROUP.

References

[1] G.G. Simmons, "Introduction to Topology and Modern Analysis", Mc Graw Hill, (1977).

[2] N. Dumford and J. Schwartz, "Linear Operators" (part I), Interscience, NY, 1958.

[3] Shlomo Sternberg, "Lectures Differential Geometry", Chelsea Publishing Company, New York, NY, (1983).