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The Atiyah-Singer Index Theorem: A Heat Kernel (PDE's) Proof

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Abstract

We present a simple Partial Differential Equation proof for the Atiyah-Singer Index Theorem in the context of Dirac Operators on a Riemman Surface. Addintionaly, we present a proof of the monotonic grown under the Ricci flux of the Dirac Operator in the presence of a abelian Gauge Connection in a Riemman Surface.

1 The Atiyah-Singer Index Theorem

Dirac type operators in Riemannian manifolds are mathematical objects of cornerstone importance in Differential Geometry ([1], [2], [3]) and with a growing usefulness in Quantum Physics ([4], [5]). Unfortunately, many of the studies related to theory of Dirac Operators are very sophisticated for applied scientists due to the use of sophisticated and cumbersome methods of Algebraic Topology and Bundles Manifolds ([1]).

In this note, we intend to present in a relatively simple way and based in the elementary aspects of the Seeley Theory of Pseudo-Differential Operators, one of the most celebrated Differential Topological Theorem: The Atiyah-Singer Index Theorem which (in one of its "palatable" version) says roughly that the trace of the evolution operator associated to a certain class of Dirac Operators defined in two-dimensional orientable compact Riemannian manifolds possess a manifold topological index – its Euler-Poincaré gennus.

Let us thus start with A denoting an elliptic differential operator of second order acting in the space $C_c^\infty(R^2)_{q\times q}$ formed by all these functions infinitely differentiable with compact support in R^2 and taking values in $M_{q\times q}(C)$ (the vector space of the complex matrices $q\times q$)

$$A=\sum_{|\alpha|\leq 2} a_\alpha(x)D_x^\alpha \tag{1}$$

where $\alpha=(\alpha_1,\alpha_2)$ are non-negative integers associated to the basic self-adjoint differential operators

$$D_x^\alpha=\left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{1}{i}\frac{\partial}{\partial x_2}\right)^{\alpha_2} \tag{2}$$

with $A_\alpha(x)\in C_c^\infty(R^2)_{q\times q}$.

Let us now consider the (contractive) semi-group generated by the operator A through the spectral calculus for operators in Banach spaces

$$\exp(-tA)=\frac{1}{2\pi i}\oint_{\widehat{C}} d\lambda\, e^{-t\lambda}\big(\lambda\,\mathbf{1}_{\varepsilon\times\varepsilon}-A\big)^{-1} \tag{3}$$

where $\widehat{C}$ is a given closed curve containing in its interior the spectrum of A: $\sigma(A)$ which is supposed to be in the semi-line R^+.

According to Seeley ([1],]2]); let us consider the operational symbol of the operator $\big(\lambda\,\mathbf{1}_{q\times q}-A\big)$ which is defined by the relationship below

$$\begin{aligned}\sigma(A-\lambda\mathbf{1})(\xi)^{\rm def}&=e^{-ix\xi}(A-\lambda\mathbf{1})e^{ix\xi}=\left(\sum_{|\alpha|\leq 2}A_\alpha(x)(\xi_1)^{\alpha_1}(\xi_2)^{\alpha_2}\right)-\lambda\,\mathbf{1}_{q\times q}\\ &=\sum_{|j|=0}A_j(x,\xi,\lambda)\end{aligned} \tag{4}$$

where for $0\leq j<2$

$$A_j(x,\xi,\lambda)=\sum_{|\alpha|=\alpha_1+\alpha_2=j}A_\alpha(x)(\xi_1)^{\alpha_1}(\xi_2)^{\alpha_2} \tag{5-a}$$

and for $j = 2$

$$A_2(x,\xi,\lambda) = -\lambda\,\mathbf{1}_{q\times q} + \left(\sum_{|\alpha|=\alpha_1+\alpha_2=2} A_\alpha(x)(\xi_1)^{\alpha_1}(\xi_2)^{\alpha_2}\right). \tag{5-b}$$

Let us now consider the usual Fourier Transforms in $L^1(R^2)\cap L^2(R^2)$ with its operational rule

$$\widehat{F}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{R^2} d^2x\, e^{i\xi x}\, f(x) \tag{6-a}$$

$$(Af)(x) = \frac{1}{(2\pi)^2}\int_{R^2} d^2\xi\, e^{ix\xi}\big[\sigma(A)(x,\xi)\big]\,\widehat{F}(\xi) \tag{6-b}$$

At this point it is of a crucial importance to note that the functions $a_j(x,\xi,\lambda)$ are all homogeneous functions of degree j when considered as functions of the variable ξ and $\sqrt{\lambda}$

$$a_j(x,c\xi,c^2\lambda) = (c^j)a_j(x,\xi,\lambda). \tag{7}$$

Now it is an important technical theorem of Seeley that the Green Function of the operator $(A-\lambda\mathbf{1})$ has a symbol which is given by a serie of smooth functions $\big\{C_{-2-j}(x,\xi)\big\}_{j=0}$ in the Seeley topology

$$\sigma\Big((A-\lambda\mathbf{1})^{-1}\Big) = \sum_{j=0}^{\infty} C_{-2-j}(x,\xi). \tag{8}$$

As a consequence of the above formulae, we have the following exactly formulae for the Inverse of any Elliptic Inversible operator A

$$\begin{aligned}\big(A^{-1}f\big)(x) &= \int_0^\infty dt\big(e^{-tA}f\big)(x)\\ &= \sum_{j=0}^{\infty}\left\{\frac{1}{2\pi i}\int d^2\xi\, d^2x\, d^2y\, e^{i\xi(x-y)}\left[\oint_{\widehat{C}} \frac{C_{-2-j}(x,\xi,\lambda)}{\lambda}\,d\lambda\right] f(y)\right\}\end{aligned} \tag{9}$$

A crucial observation can be made at this point. One can introduce a non-commutative multiplicative operation among the Seeley symbols associated to Elliptic Operators A and B acting on the domain $C_c^\infty(R^2)_{q\times q}$. Namely

$$\sigma(A)\circ\sigma(B) \overset{\text{def}}{\equiv} \sigma(A\circ B) \tag{10}$$

where $A\circ B$ is the operator composition. Explicitly, we have that

$$\sigma(A)\circ\sigma(B) = \sum_{|\alpha|=\alpha}\big[D_\xi^\alpha\,\sigma(A)(x,\xi)\big]\big[i\,D_x^\alpha\,\sigma(B)(x,\xi)\big]/\alpha! \tag{11}$$

Here

$$\alpha! = \alpha_1! \cdot \alpha_2! \tag{12}$$

From the obvious relationship

$$\big(A - \lambda\mathbf{1}\big) \circ \big(A - \lambda\mathbf{1}\big)^{-1} = 1 \tag{13}$$

we have that

$$\sigma\big(A - \lambda\mathbf{1}\big) \circ \sigma\Big(\big(A - \lambda\mathbf{1}\big)^{-1}\Big) = 1. \tag{14}$$

As a consequence it yields the result

$$\left\{\frac{1}{\alpha!}\sum_{|\alpha|\leq 2}\big(D_\xi^\alpha\,\sigma(A-\lambda\mathbf{1})(x,\xi)\big)i\,D_x^\alpha\left(\sum_{j=0}^{\infty}C_{-2-j}(x,\xi)\right)\right\} = 1 \tag{15}$$

or in on equivalent way

$$\frac{1}{\alpha!}\sum_{|\alpha|\leq 2}\left\{\sum_{j=0}^{\infty}\big[\big(D_\xi^\alpha\,\sigma\big(A-\lambda\mathbf{1}\big)(x,\xi)i\,D_\xi^\alpha\big(C_{-2-j}(x,\xi)\big)\big]\right\} = 1. \tag{16}$$

The above written equations can be solved by introducing the scaled variables ($t > 0$)

$$\xi = t\,\xi' \;;\; \lambda^{1/2} = t(\lambda')^{1/2} \tag{17}$$

and

$$C_{-2-j}\left(x, t\xi', \big(\big(t\lambda'\big)^{\frac{1}{2}}\big)^2\right) = t^{-(2+j)}\,C_{-2-j}(x,\xi',\lambda'). \tag{18}$$

After substituting eq.(18) into eq.(16) and by comparing the resulting power series in the scale factor $1/t$, one obtains the famous Seeley recurrence equations which for $t \longrightarrow 1$, give us explicitly expressions of all Seeley coefficients $\{C_{(-2+j)}(x,\xi)\}$ of the Green Function of the operator A (note that they are elements of $M_{q\times q}(C)$)

$$C_{-2}(x,\xi) = \big(a_2(x,\xi)\big)^{-1} \tag{19}$$

$$0 = a_2(x,\xi)C_{-2-j}(x,\xi) + \frac{1}{\alpha!}\left(\sum_{\substack{\ell<j \\ k-|\alpha|-2-\ell=-j}}(D_\xi^\alpha\,a_k(x,\xi))(i\,D_x^\alpha\,C_{-2-\ell}(x,\xi))\right) \tag{20}$$

For the Laplacian like elliptic operator below:

$$
\begin{aligned}
A = &-\left(g_{11}(x_1,x_2)\,\frac{\partial^2}{\partial x_1^2} + g_{22}(x_1,x_2)\frac{\partial^2}{\partial x_2^2}\right)\mathbf{1}_{q\times q} \\
&+ \big(A_1(x_1,x_2)\big)_{q\times q}\,\frac{\partial}{\partial x_1} + \big(A_2(x_1,x_2)\big)_{q\times q}\,\frac{\partial}{\partial x_2} \\
&+ \big(A_0(x,x_2)\big)_{q\times q}
\end{aligned}
\tag{21}
$$

where $\left\{g_{11}(x), g_{22}(x); x\in R^2\right\}$ are positive definite functions in $C_c^\infty(R^2)_{q\times q}$, we can evaluate exactly the Seeley relationship eq.(20).

Now it is a tedious evaluation to see that (important exercise left to our readers) we have the result

$$a_2(x,\xi,\lambda) = \big(g_{11}(x)\xi_1^2 + g_{22}(x)\xi_2^2 - \lambda\,\mathbf{1}\big) \tag{22-a}$$

$$a_1(x,\xi,\lambda) = -i\,A_1(x)\xi_1 - i\,A_2(x)\xi_2 \tag{22-b}$$

$$a_0(x,\xi,\lambda) = -A_0(x) \tag{22-c}$$

$$C_{-2}(x,\xi,\lambda) = \big(g_{11}(x)\xi_1^2 + g_{22}(x)\xi_2^2 - \lambda\,\mathbf{1}\big)^{-1} \tag{22-d}$$

$$
\begin{aligned}
c_{-3}(x,\xi,\lambda) = &\; i\big(A_1(x)\xi_1 + A_2(x)\xi_2\big)\big(C_{-2}(x,\xi,\lambda)\big)^2 \\
&- 2i\,g_{11}(x)\xi_1\left[\left(\frac{\partial}{\partial x_1}\,g_{11}\right)(x)\,\xi_1^2 + \left(\frac{\partial}{\partial x_1}\,g_{22}(x)\right)\xi_2^2\right] \times \big(C_{-2}(x,\xi,\lambda)\big)^3 \\
&- 2i\,g_{22}\xi_2\left[\left(\frac{\partial}{\partial x_2}\,g_{11}(x)\right)(\xi_1)^2 + \left(\frac{\partial}{\partial x_2}\,g_{22}(x)\right)(\xi_2)^2\right]\big(C_{-2}(x,\xi,\lambda)\big)^3
\end{aligned}
\tag{22-e}
$$

We now analyse the symbolic-operational Seeley expansion for the evolution semi-group eq.(3)

$$
\begin{aligned}
&\mathrm{Tr}_{(C_c^\infty(R^2))_{q\times q}}\big[\exp(-tA)\big] \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi}\right)^2 \int_{R^2} d^2x\, \sigma(\exp(-tA))(x,\xi) \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi}\right)^2 \frac{1}{2\pi i}\int_{+\infty}^{-\infty} d(is)e^{ist}\int_{R^2\times R^2} d^2x\, d^3x\, C_{-2-j}(x,\xi,-is) \\
&= \sum_{j=0}^{\infty}\left\{\frac{1}{t}\,\frac{1}{(2\pi)^3}\int_{R^2} d^2\xi \int_{R^2} d^2x \int_{-\infty}^{+\infty} e^{is}\, C_{-2-j}\left(x,\xi,\frac{-is}{t}\right)ds\right\} \\
&= -\left\{\sum_{j=0}^{\infty}\frac{1}{t}\,\frac{1}{(2\pi)^3}\int_{R^2\times R^2} d^2x\, d^2\xi \int_{-\infty}^{+\infty} e^{is}(t)^{(\frac{2+j}{2})}C_{-2-j}(x,t^{1/2}\xi,-is)\right\} \\
&= \sum_{j=0}^{\infty}\left\{t^{\frac{(j-2)}{2}}\,(2\pi)^3\int_{R^2} d^2\xi \int_{R^2} d^2x \int_{-\infty}^{+\infty} ds\, C_{-2-j}(x,\xi,-is)\right\} \qquad (23)
\end{aligned}
$$

After substituting eqs.(22) (together with the term $C_{-4}(x,\xi,\lambda)$ not writte here), we have the Seeley short-time expansion for the Heat-Kernel evolution operator associated to our Laplacean A as given by eq.(21).

$$
\begin{aligned}
&\mathrm{Tr}_{C_c^\infty(R^2)_{q\times q}}\big[\exp(-tA)\big] \\
&\overset{t\to 0^+}{\sim}\ \frac{1}{4\pi t}\left(\int_{R^2} d^2x\,(\sqrt{g_{11}\,g_{22}})\,(x)\right)\mathbf{1}_{q\times q} \\
&+\frac{1}{4\pi}\left(\int_{R^2} d^2x\,\sqrt{g_{11}\,g_{22}}\left(-\frac{1}{6}\,R(g)\right)\right)\mathbf{1}_{q\times q} \\
&-\frac{1}{2}\times\frac{1}{\sqrt{g_{11}\,g_{22}}}\left[\left(\frac{\partial}{\partial x_1}(\sqrt{g_{11}\,g_{22}}\cdot A_1)\right)+\left(\frac{\partial}{\partial x_1}(\sqrt{g_{11}\,g_{22}}\cdot A_1)\right)\right](x) \\
&-\frac{1}{4}\,\big[\tilde g_{11}(A_1)^2+\tilde g_{33}(A_2)^2+A_0\big]\,(x)+O(t) \qquad (24)
\end{aligned}
$$

Here $R(g)$ is the curvature scalar associated to the metric tensor $ds^2 = g_{11}(dx_1)^2 + g_{22}(dx_2)^2$. The inverse metric is denoted by $\{\tilde g_{11},\tilde g_{22}\}$.

Let us give a proof of the famous Atiyah-Singer index theorem in the context of the Heat kernel PDE's techniques.

Let us thus consider complex coordinates in a given (bounded) open subset $W \subset R^2$, supposed to be a chart of a given Riemann Surface $\mathcal{M}$.

$$\begin{aligned} z &= x_1 + i\,x_2 \qquad ; \qquad & \frac{\partial}{\partial z} &= \frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2} \\ \overline{z} &= x_1 - i\,x_2 & \frac{\partial}{\partial \overline{z}} &= \frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2} \end{aligned}$$

For each integer j, let us introduce Hilbert Spaces $\mathcal{H}_1$ and $\overline{\mathcal{H}}_j$ defined as ([5])

$$\mathcal{H}_j = \begin{cases} \text{vectorial set of all complex valued functions which under the} \\ \text{action of a complex coordinate transformation } z = z(w) \text{ of } W, \\ \text{has the tensorial like transformation law} \\ \quad f(z,\overline{z}) = (\partial w/\partial z)^{-j}\,\tilde{f}(w,\overline{w}) \end{cases} \tag{26}$$

We now introduce a Hilbertian Structure in $\mathcal{H}_j$ by the following inner product

$$(g,f)_{\mathcal{H}_j} = \int_{R^2} dz\,d\overline{z}\big(\rho(z,\overline{z})\big)^{j+1} f(z,\overline{z})\,\overline{g(z,\overline{z})} \tag{27}$$

where $\rho(z,\overline{z})$ denotes a real continuous function in W and provenient of a conformal metrical structure in a given Riemann Surface with local chart W

$$ds^2 = \rho(z,\overline{z})dz \wedge d\overline{z}. \tag{28}$$

The Hilbert Space $\overline{\mathcal{H}}_j$ is defined in an analogous way to the definition eq.(26), but with the dual tensor transformation law

$$f(z,\overline{z}) = \left(\big(\overline{\partial w/\partial z}\big)\right)^{-j}\,\tilde{f}(w,\overline{w}). \tag{29}$$

As an exercise to our readers we have that the above inner products are invariant under the action of the conformal (complex) coordinates transformations of the open set W.

Let us now introduce the following self-adjoint operators in $L^2(\overline{W})$;

$$\begin{aligned} L_j\colon \mathcal{H}_j &\longrightarrow \overline{\mathcal{H}}_{-(j+1)} \\ f &\longrightarrow \rho^j\,\partial_{\overline{z}}\,f \end{aligned} \tag{30}$$

and its adjoint operator

$$\begin{aligned} L_j^*\colon \overline{\mathcal{H}}_{-(j+1)} &\longrightarrow \mathcal{H}_(j \\ f &\longrightarrow \rho^{-(j+1)}\,\partial_z\,f \end{aligned} \tag{31}$$

We consider the further positive definite (inversible) self-adjoint operators

$$\mathcal{L}_j\colon L_j^* \, L_j \colon \mathcal{H}_j \longrightarrow \mathcal{H}_j \tag{32-a}$$

$$\mathcal{L}_j^*\colon L_j \, L_j^+ \colon \overline{\mathcal{H}}_{-(j+1)} \longrightarrow \overline{\mathcal{H}}_{-(j+1)} \tag{32-b}$$

Note the explicitly expression:

$$\mathcal{L}_j \, \mathcal{L}_j^* = \big(-\rho^{-1}\,\partial_{\overline{z}}\partial_z\big) + \big((j+1)\rho^{-1}\big(\partial_{\overline{z}}\big(\ell g\,\rho\,\partial_z\big)\big)\big) \tag{33-a}$$

$$\mathcal{L}_j^* \, \mathcal{L}_j = \big(-\rho^{-1}\,\partial_z\partial_{\overline{z}}\big) - \big(j\,\rho^{-1}\big(\partial_z\,\ell g\,\rho\,\partial_{\overline{z}}\big)\big) \tag{33-b}$$

By using now the Seeley asymptotic expansion in $C_c^\infty(R^2)$, we have the following $t \to 0^+$ expansions for the Heat Kernels associated to the Dirac operator eq.(33) ([5])

$$\begin{aligned} &\lim_{t\to 0^+} \ \mathrm{Tr}_{C_c^\infty(R^2)_{2\times 2}}\big(\exp\big(-t\,\mathcal{L}_j\mathcal{L}_j^*\big)\big) \\ &= \int_W \left(\frac{dz\wedge d\overline{z}}{2i}\right)\left(\frac{\rho(z,\overline{z})}{2\pi t} - \frac{(1+3j)}{12\pi}\,\big(\Delta\ell g\,\rho\big)(z,\overline{z})\right) + O(t) \end{aligned} \tag{34-a}$$

$$\begin{aligned} &\lim_{t\to 0^+} \ \mathrm{Tr}_{C_c^\infty(R^2)_{2\times 2}}\big(\exp\big(-t\,\mathcal{L}_j^*\mathcal{L}_j\big)\big) \\ &= \int_W \frac{dz\wedge d\overline{z}}{2i}\left(\frac{\rho(z,\overline{z})}{2\pi t} + \frac{(2+3j)}{12\pi}\,\big(\Delta\ell g\,\rho\big)(z,\overline{z})\right) + O(t) \end{aligned} \tag{34-b}$$

Now the famous topological Heat Kernel index can be obtained easily through the Atiyah-Singer definition.

$$\begin{aligned} \mathrm{index}(\mathcal{L}_j) &= \lim_{j\to 0^+}\big[\mathrm{Tr}_{C_c^\infty(R^2)_{2\times 2}}\left[\exp(-t\mathcal{L}_j\mathcal{L}_j^*) - \exp(-t\mathcal{L}_j^*\mathcal{L}_j)\right]\big] \\ &= \frac{(1+2j)}{4\pi}\left[\int_W \frac{dz\wedge d\overline{z}}{2\pi i}\,\rho(z,\overline{z})\left(+\frac{1}{\rho(z,\overline{z})}\,\Delta\ell g\,\rho(z,\overline{z})\right)\right] \\ &= \left(\frac{1+2j}{4\pi}\right)\chi(\mathcal{M}) \end{aligned} \tag{35}$$

Here the curvature of the Riemann Surface $\mathcal{M}$ with metric $ds^2 = \rho(z,\overline{z})\,dz\wedge d\overline{z}$ is given by

$$R(z,\overline{z}) = \left(\frac{1}{\rho}\,\Delta\,\ell g\,\rho\right)(z,\overline{z}) \tag{36}$$

which is related to the topological invariant of the Euler-Poincaré genus of $\mathcal{M}$ through the Gauss theorem

$$\int_{\mathcal{M}} \rho(z,\overline{z})R(z,\overline{z}) = 2\pi(2-2g) \tag{37}$$

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APPENDIX 1

Normalized Ricci Fluxes in Closed Riemann Surfaces and the Dirac Operator in the Presence of an Abelian Gauge Connection

In this short appendix 1 we want to expose a new result of ours on the Riemannian Geometry of Riemann Suraces $\mathcal{M}$. Let us thus consider the non linear parabolic PDE's equation governing the flux dynamics of a given metric $h_{\mu\nu}(x)$ in $\mathcal{M}$. Namely

$$\frac{\partial}{\partial t} h_{\mu\nu} = (R_0 - R)h_{\mu\nu} \tag{1}$$

where R is a scalar of curvature and R_0 its mean value.

An important result in this subject of Ricci Fluxes is the famous theorem of Osgood-Phillips-Sarnak, which show that in the class of all metrical structures with a fixed conformal structure, the determinant of the Laplace Operator takes its maximal value in the metric of constant curvature ([1]). Let us present a generalization of such result for a Dirac Operator in the presence of an Abelian Gauge Connection (with a fixed spin structure).

Theorem. The determinant of the Dirac Operator in the presence of a Abelian Gauge connection in a Compact Orientable Riemann Manifold (equivalent to a Complex Curve = Riemann Surface) has a monotonic grown under the Ricci flux.

Proof: Let us consider a Dirac Operator with a spin structure (v^i, u^i) in the presence of a $U(1)$ Abelian gauge connection which in the Physicists tensor notation reads as of as ([2])

$$\not{D}(A, \hat{h}) = i\,\gamma^a\,\hat{e}^\mu_a \left(\partial_\mu + \frac{1}{8}\,W_{\mu,ab}(\hat{e})\,\varepsilon^{ab}\,\gamma_5 + A_\mu\right). \tag{2}$$

The matrices $\gamma^\mu = \hat{e}^\mu_a\,\gamma_a$ are the (euclidean) Dirac matrices associated to the metrical Riemann surface structure $\hat{h}^{\mu\nu} = \hat{e}^\mu_a\,\hat{e}^\nu_a$. Note that $\hat{h}^{\mu\nu}$ can be always written into the canonical conformal form ($\zeta \in \mathcal{M}$)

$$\hat{h}^{\mu\nu}(\zeta) = \frac{1}{\rho(\zeta)}\,\tilde{h}^{\mu\nu}(\zeta), \tag{3}$$

where $\tilde{h}^{\mu\nu}$ is the element representative of $\hat{h}^{\mu\nu}$ in the Teichmiller modulo space of $\mathcal{M}$. We note too that the Abelian Gauge Connection (the Hodge abelian connection) has the usual decomposition ([2])

$$A_\mu = -\frac{\varepsilon^{\mu\nu}}{\sqrt{h}}\,\partial_\mu\phi + A_\mu^H \tag{4}$$

with

$$\nabla^\mu\big(A_\mu - A_\mu^H\big) \equiv 0. \tag{5}$$

It is useful to remark that the effects of the non trivial topology of the genus of Riemann Surface $\mathcal{M}$ reflects itself in the Hodge Harmonic term A_μ^H of the $U(1)$-Connection through the Abelian differential forms (and its respectives complex conjugates) ([2])

$$A_\mu^H = 2h\left(\sum_{\ell=1}^{g}(p_\ell\,\alpha_\mu^\ell + g_\ell\,\beta_\mu^\ell)\right) \tag{6-a}$$

$$\alpha_\mu^i = -\overline{\Omega}_{ik}(\Omega-\overline{\Omega})_{ks}^{-1}\,w_\mu^f + c.c \tag{6-b}$$

$$\beta_\mu^i = (\Omega-\overline{\Omega})_{ij}^{-1}\,w_\mu^j + c.c \tag{6-c}$$

The Riemann Surface Matrix Period Ω is defined by the usual homological relationships (Abel Integrals)

$$\int_{a^j}\alpha^i = \delta_{ij} \quad ; \quad \int_{b^i}\beta^j = \Omega_{ij} \tag{7}$$

where $\{a^i\}$ and $\{b^j\}$ are the canonical homological cycles of $\mathcal{M}$.

Let us now consider the variation of the (functional) determinant of the Dirac operator $\det^{1/2}(\not{D}\,\not{D}^*) = \det(\not{D})$ in relation to an infinitesimal variation of the metric with a fixed conformal class ([3])

$$\begin{aligned}
&\frac{\partial}{\partial t}\,\ell g\left\{\frac{\det\big(\not{D}(A,\hat{h})\big)}{\mathrm{area}(\mathcal{M})\det(\not{D}(A=0,\tilde{h}))}\right\}\\
&\quad= \frac{1}{\pi}\int_{\mathcal{M}}\frac{dz\wedge d\overline{z}}{2i}\,\big[\tilde{h}^{z\overline{z}}(\partial_z\varphi\,\partial_{\overline{z}}\varphi)(z,\overline{z})\big]\\
&\quad+\frac{1}{12\pi}\int_{\mathcal{M}}\frac{dz\wedge d\overline{z}}{2i}\,\tilde{h}^{z\overline{z}}\big[(\ell g\,\rho)_t(\ell g\,\rho)_{z\overline{z}}\big].
\end{aligned} \tag{8}$$

Note that this metrical variation is evaluated for the normalized Ricci flux below

$$\frac{\partial}{\partial t}\,\ell g\,\rho \equiv [\ell g\,\rho]_t = R_0 - R \tag{9-a}$$

$$R_0 = \frac{2\pi(2-g)}{\text{area}(\mathcal{M})} \tag{9-b}$$

$$R = -\frac{1}{\rho}\,\partial_{z\bar{z}}(\ell g\,\rho). \tag{9-c}$$

As a consequence we have the positivity of the eq.(8)

$$\frac{\partial}{\partial t}\,\ell g \left\{ \frac{\det \not{D}(A,\hat{h})}{\text{area}(\mathcal{M})\,\det(\not{D}(A=0,\tilde{h}))} \right\} \geq 0 \tag{10}$$

At this point we remark that eq.(10) vanishes solely for all those metric of constant curvature, which at the asymptotic limit $t \to \infty$ leads us to the usual result ($R \to R_0$)

$$\int_{\mathcal{M}} (R_0 - R)\, R \cdot \sqrt{h}\; dz \wedge d\bar{z} \overset{t\to\infty}{\longrightarrow} 0$$

since for $t \to \infty$ all (smooth-C^{∞}) metrics on $\mathcal{M}$ are attracted to the metric of constant curvature under the action of Ricci Flux with a fixed area $\left(\int_{\mathcal{M}} \sqrt{\hat{h}}\,\frac{dz \wedge d\bar{z}}{2i} = \text{area}(\mathcal{M})\right)$.