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Generalized Dirac-Coulomb Problem

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SUSY SHAPE-INVARIANT HAMILTONIANS FOR THE GENERALIZED DIRAC-COULOMB PROBLEM

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Abstract

A spin $\frac{1}{2}$ relativistic particle described by a general potential in terms of the sum of the Coulomb potential with a Lorentz scalar potential is investigated via supersymmetry in quantum mechanics.

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Supersymmetry in Quantum Mechanics (SUSY QM) [1] is of intrinsic mathematical interest in its own as it connects otherwise apparently unrelated second-order differential equations.

The (1+3) and (1+1) dimensional Dirac equations with both scalar-like and vector-like potentials are well known in the literature for a long time [2]. The connection between position-dependent-effective-mass and shape invariant condition under parameter translation has been discussed in non-relativistic quantum mechanics [3, 4]. Recently, some relativistic shape invariant potentials have been investigated [5].

Exact solutions for the bound states in this mixed potential can be obtained by the method of separation of variables [6–8] and also by the use of the dynamical algebra $SO(2, 1)$ [9]. In a recent paper the solution of the scattering problem for this potential has been obtained by an analytic method and also by an algebraic method [10], the problem of a relativistic Dirac electron with a $1/r$ scalar potential, as well as a Dirac magnetic monopole and an Aharonov-Bohm potential has also been investigated [11], and the bound eigenfunctions and spectra of a Dirac hydrogen atom have been found via $su(1, 1)$ Lie algebra [12].

Recently exact solutions have been found for fermions in the presence of a classical background which is a mixing of the time-dependent of a gauge potential and a scalar potential [13]. Also, exactly solvable Eckart scalar and vector potentials in the Dirac equation have been investigated via SUSY QM [14], the S -wave Dirac equation has been solved exactly for a single particle with spin and pseudospin symmetry moving in a central Woods-Saxon potential [15].

The special case of the non-relativistic [16] and relativistic Coulomb problems have been treated recently via SUSY QM [17–19]. In this work, the relativistic Coulomb potential with a Lorentz scalar potential is investigated via shape invariance conditions of the SUSY QM.

The time independent Dirac equation may be written in the form $H\Psi = E\psi$, where the Hamiltonian is given by

$$H = \rho_1 \otimes \vec{\sigma} \cdot \vec{p} + \left(M - \frac{A_2}{r} \right) \rho_3 \otimes \mathbf{1}_{2 \times 2} - \frac{A_1}{r} \otimes \mathbf{1}_{4 \times 4},$$

and we have used a direct product notation in which ρ_i and σ_i , ($i = 1, 2, 3$) are the Pauli spin matrices obeying $[\rho_i, \sigma_j]_- = 0$, with $\hbar = c = 1$.

We consider [20]

$$\Psi = \begin{pmatrix} \frac{iG_{\ell j}}{r} \phi_{jm}^{\ell} \\ \frac{F_{\ell j}}{r} \vec{\sigma} \cdot \vec{n} \phi_{jm}^{\ell} \end{pmatrix}, \quad (1)$$

where $\phi_{jm}^\ell = \phi_{jm}^{(\pm)}$, for $j = \ell \pm \frac{1}{2}$. Next, using the relation $[\mathbf{1} + \vec{\sigma} \cdot \vec{L}, \vec{\sigma} \cdot \vec{n}]_+ = 0$ we obtain $K\Psi = -k\Psi$ and the following radial equations

$$\begin{aligned} \frac{dG_{\ell j}}{dr} + \frac{k}{r}G_{\ell j} - \left(E + M - \frac{A_2}{r} + \frac{A_1}{r}\right)F_{\ell j} &= 0, \\ \frac{dF_{\ell j}}{dr} - \frac{k}{r}F_{\ell j} + \left(E - M + \frac{A_2}{r} + \frac{A_1}{r}\right)G_{\ell j} &= 0. \end{aligned} \quad (2)$$

Note that the interaction in these two equations can be diagonalized so that we obtain

$$A^+\hat{G} \propto \hat{F}, \quad A^-\hat{F} \propto \hat{G} \quad (3)$$

where

$$A^\pm = \pm \frac{d}{dr} + \frac{\lambda}{r} - \frac{EA_1 + MA_2}{\lambda}. \quad (4)$$

These relations are similar to the relations between the two components of the eigenfunctions of a "supersymmetric" Hamiltonian which satisfies the following Lie graded algebra

$$\mathcal{H} = [\mathbf{Q}, \mathbf{Q}^\dagger]_+ = \mathbf{Q}\mathbf{Q}^\dagger + \mathbf{Q}^\dagger\mathbf{Q}, \quad [\mathcal{H}, \mathbf{Q}^\dagger]_- = 0 = [\mathcal{H}, \mathbf{Q}]_- \quad (5)$$

with the following representation

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} H_- = A^+A^- & 0 \\ 0 & H_+ = A^-A^+ \end{pmatrix}, \quad \Phi_{SUSY} = \begin{pmatrix} F \\ G \end{pmatrix}. \quad (6)$$

Note that the supercharges are nilpotent operators, viz., $(\mathbf{Q}^\dagger)^2 = 0 = \mathbf{Q}^2$.

Thus, using the shape invariant Hamiltonians H_\pm we obtain the energy eigenvalues associated to the component \hat{F}^n given by

$$E_n = \sqrt{\frac{M^2}{1 + \frac{\gamma_n^2}{(\sqrt{k^2 - \gamma_n^2 + n})^2}}} \quad n = 0, 1, 2, \dots, \quad \gamma_n(E) = A_1 + \frac{MA_2}{E_n}. \quad (7)$$

In conclusion, we obtain the complete set of the energy eigenvalues of the Dirac equation for a potential which is the sum of the Coulomb potential with a Lorentz scalar potential inversely proportional to r via shape invariance property as applied in [17]. One of us (RLR) will make elsewhere a detailed analysis for this problem as applied to the relativistic Coulomb potential via SUSY shape-invariant potentials [17].

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